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On the structure of equations integrable by the arbitrary-order linear spectral problem

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Abstract. The general form of partial differential equations integrable by the arbitrary-order linear spectral problem is found. The groups of Bäcklund transformations corresponding to these equations are constructed. It is shown that partial differential equations of the class under study are Hamiltonian ones. Some reductions of general equations are considered. In particular, the Hamiltonian structure of the generalisations of the sine-Gordon equation to the groups GL(N), SU(N) and SO(N) at arbitrary N is proved.

1. Introduction

The inverse spectral transform (IST) method allows a comprehensive study of a great number of various partial differential equations (see e.g. Scott *et al* (1973), Calogero (1978), Bullough and Caudrey (1980)). The general scheme of this method was discussed in Zakharov and Shabat (1974a,b) and Zakharov and Mikhailov (1978).

All the differential equations to which the IST method is applicable are united in the classes of equations integrable by the same linear spectral problem. A simple and convenient description of the class of equations which are integrable with the help of the second-order linear (in the spectral parameter) spectral problem was presented in Ablowitz *et al* (1974). This class of equations is characterised by the $(n-1)$th arbitrary function (n is the number of independent variables) and by a certain integro-differential operator (Ablowitz *et al* 1974, Calogero and Degasperis 1976). Analogous results were obtained for the class of equations which are associated with the matrix stationary Schrödinger equation (Calogero and Degasperis 1977), the general linear spectral problem of arbitrary order (Newell 1979, Kulish 1979, Konopelchenko 1979a), with the second-order linear problem quadratic in its spectral parameter (Gerdjikov *et al* 1979) and with the general arbitrary-order linear spectral problem polynomial with respect to the spectral parameter (Konopelchenko 1979b). Within the framework of this approach the wide classes of Bäcklund transformations (BTS), which play a significant role in a study of nonlinear differential equations, have also been found (Calogero and Degasperis 1976, 1977, Konopelchenko 1979a,b). For equations integrable by the second-order linear problem, the Hamiltonian structure of all the equations of this class is analysed (Flaschka and Newell 1975).

In the present paper we are going to study a class of partial differential equations connected with the general linear spectral problem of arbitrary order:

$$\partial\psi/\partial x = (\mathrm{i}\lambda A + \mathrm{i}P(x, t, \ldots))\psi \tag{1.1}$$

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where λ is the spectral parameter, A the constant diagonal matrix ($A_{ik} = a_i\delta_{ik}$, $a_i \neq a_k$, $i, k = 1, \ldots, N$) and the 'potentials' $P(x, t, \ldots)$ are $N \times N$ matrices. It is not assumed here, in contrast to Newell (1979) and Konopelchenko (1979a), that $P_{ii} = 0$ ($i = 1, \ldots, N$).

We find the general form of equations integrable by means of equation (1.1) and construct the Bäcklund transformations corresponding to these equations. As will be seen, BTs and integrable equations are closely connected with the group of transformations conserving the form of the spectral problem (1.1).

It is shown in this paper that the equations integrable with the help of equation (1.1) are Hamiltonian ones, both in the general case and in the cases when the 'potentials' satisfy the relations $P^+ = P$ and $P_T = -P$. The case of the singular dispersion law is also studied. Among such equations there are the relativistic-invariant equations coinciding with the non-Abelian generalisations of the sine-Gordon equation (Budagov and Takhtajan 1977, Budagov 1978), which, as is known, are gauge equivalent to the equations of the principal chiral field (Budagov 1978, Zakharov and Mikhailov 1978). The Hamiltonian structure of these equations for the groups GL(N), SU(N) and SO(N) with arbitrary N is proved. For all equations considered in this paper, uniqueness of the symplectic structure occurs (similarly to the case $N = 2$ (Magri 1978, Kulish and Reiman 1978)).

We mainly use the compact matrix notation proposed in Newell (1979). Let us recall it: for an arbitrary matrix Q the matrices Q_F and Q_D are determined as follows:

$$(Q_F)_{ik} = Q_{ik} \quad \text{for } i \neq k \ (i, k = 1, \ldots, N), \qquad (Q_F)_{ii} = 0 \ (i = 1, \ldots, N),$$
$$(Q_D)_{ik} = 0 \quad \text{for } i \neq k \ (i, k = 1, \ldots, N), \qquad (Q_D)_{ii} = Q_{ii} \ (i = 1, \ldots, N).$$

The matrix Q_R is given by the relation $[A, Q_R] = Q$, i.e.

$$(Q_R)_{ik} = \frac{1}{a_i - a_k} Q_{ik} \qquad (i \neq k; i, k = 1, \ldots, N).$$

The paper is organised as follows. The form of transformations of the transition matrix and potentials P conserving the spectral problem (1.1) is found in § 2. The way in which the integrable equations and Bäcklund transformations are connected with these transformations is shown in § 3. The equations with singular dispersion law and, in particular, the relativistic-invariant equations are considered in § 4. The fifth section is devoted to the Hamiltonian structure of integrable equations with $P \in$ algebra GL(N) and $P \in$ algebra SU(N).

Equations with $P_T = -P$ are examined in § 6. Their Hamiltonian structure is proved in the last, seventh section. In particular, the Hamiltonian structure of the generalisation of the sine-Gordon equation to the group SO(N) with arbitrary N is proved, and the explicit form of the Hamiltonian is found.

2. The group of transformations conserving the linear spectral problem

Let us examine an arbitrary transformation $P \to P'$, $\psi \to \psi'$, which conserves the mapping $P(x, t) \leftrightarrow \psi(x, t, \lambda)$ given by the set of linear differential equations (1.1).

It is easy to see that

$$\psi' - \psi K = -\mathrm{i}\psi \int_x^\infty \mathrm{d}y \, \psi^{-1}(P' - P)\psi' \tag{2.1}$$

where the constant matrix K is determined by the asymptotic properties of the matrix solutions ψ.

It is assumed that $P(x, t, \ldots) \to 0$ as $|x| \to \infty$†. Then $\psi \to E = \exp(\mathrm{i}\lambda Ax)$ as $|x| \to \infty$. Let us introduce, following Zakharov and Manakov (1975), fundamental matrix solutions F^+, F^- with asymptotics $F^\pm \to E$ as $x \to \pm\infty$ and the transition matrix S: $F^+(x, t, \lambda) = F^-(x, t, \lambda)S(\lambda, t)$.

Setting $\psi = F^+$ and going to the limit $x \to -\infty$ in equation (2.1), we obtain

$$S' - S = -\mathrm{i}S \int_{-\infty}^{+\infty} \mathrm{d}x\, \overset{-1}{F^+}(P' - P)F^{+\prime}. \tag{2.2}$$

Formula (2.2), which relates a change of the potentials P to a change of the transition matrix, is a basis for our further discussion.

Let us suppose that the transition matrix at $P \to P'$, $\psi \to \psi'$ is transformed as

$$S \to S' = B^{-1}SC \tag{2.3}$$

where B and C are diagonal matrices independent of the variable x. Rewriting equation (2.3) in the form $S' - S = (1 - B)S' - S(1 - C)$ and comparing it with equation (2.2), we have

$$\{S^{-1}(1 - B)S'\}_F = -\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}x\, \{\overset{-1}{F^+}(P' - P)F^{+\prime}\}_F, \tag{2.4}$$

$$\{S^{-1}(1 - B)S'\}_D - 1 + C = -\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}x\, \{\overset{-1}{F^+}(P' - P)F^{+\prime}\}_D. \tag{2.5}$$

It follows from formulae (2.4) and (2.5) that transformations (2.3) are given by the matrix B which can be arbitrary. Matrix C is determined by equality (2.5).

Taking into account the relation

$$\begin{aligned}\{S^{-1}(1 - B)S'\}_F &= -\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\partial}{\partial x} \{\overset{-1}{F^+}(1 - B)F^{+\prime}\}_F \\ &= \mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}x\, \{\overset{-1}{F^+}[P(1 - B) - (1 - B)P']F^{+\prime}\}_F\end{aligned}$$

we obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \{\overset{-1}{F^+}(BP' - PB)F^{+\prime}\}_F = 0. \tag{2.6}$$

Rewriting equation (2.6) by components and introducing the notation $\overset{++}{\tilde{\varphi}}{}^{(in)}_{kl} = (\overset{-1}{F^+})_{ik}(F^{+\prime})_{ln}$, we have $(B_{ik}(\lambda) = B_i(\lambda)\delta_{ik})$

$$\int_{-\infty}^{+\infty} \mathrm{d}x \sum_{k,l=1}^{N} (B_k(\lambda)P'_{kl}(x, \ldots) - B_l(\lambda)P_{kl}(x, \ldots))\overset{++}{\tilde{\varphi}}{}^{(in)}_{kl}(x, \ldots, \lambda) = 0$$
$$(i \neq n;\ i, n = 1, \ldots, N). \tag{2.7}$$

Formula (2.7) contains the product $B(\lambda)\varphi^{(in)}(\lambda)$ which is given in a local manner, at each point λ of the bundle (1.1). The spectral problem (1.1) makes it possible to

† The case $P(x) \to P_0$ as $|x| \to \infty$, where P_0 is the constant diagonal matrix, is reduced to this by the transformations $\psi \to \exp(-\mathrm{i}P_0x)\psi$ and $P \to \exp(-\mathrm{i}P_0x)(P - P_0)\exp(\mathrm{i}P_0x)$.

transform this local product into the global product determined already on the whole bundle.

As shown in appendix 1, in a space which covers all the non-diagonal quantities $\overset{++}{\tilde{\varphi}}{}^{(in)}_{kl}$ $(i \neq n,\ k \neq l;\ i, k, l, n = 1, \ldots, N)$, the relation

$$\Lambda_R \overset{++}{\tilde{\varphi}}{}^{(in)}_F = \lambda \overset{++}{\tilde{\varphi}}{}^{(in)}_F, \qquad i \neq n;\ i, n = 1, \ldots, N, \tag{2.8}$$

holds where

$$\begin{aligned}\Lambda\varphi = {} & \mathrm{i}\,\partial\varphi/\partial x + (\varphi P'_T(x) - P_T(x)\varphi)_F \\ & - \mathrm{i}\Delta^{-1}(x) \int_x^{\infty} \mathrm{d}y\ \Delta(y)(\varphi(y)P'_T(y) - P_T(y)\varphi(y))_D P'_{TF}(x) \\ & + \mathrm{i}P_{TF}(x)\Delta^{-1}(x) \int_x^{\infty} \mathrm{d}y\ \Delta(y)(\varphi(y)P'_T(y) - P_T(y)\varphi(y))_D.\end{aligned} \tag{2.9}$$

Here and below P_T stands for the transposed matrix P. Thus, for entire functions $B_k(\lambda)$,

$$\sum_{\substack{q,p \\ (q \neq p)}} \{B_k(\Lambda_R)\}_{klqp} \overset{++}{\tilde{\varphi}}{}^{(in)}_{qp} = B_k(\lambda) \overset{++}{\tilde{\varphi}}{}^{(in)}_{kl}. \tag{2.10}$$

By virtue of this, equality (2.7) may be written as (one must extract the contribution of diagonal quantities $\overset{++}{\tilde{\varphi}}{}^{(in)}_{kk}$ and take into account equation (A1.3))

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x \sum_{\substack{q,p \\ (q \neq p)}} \Bigg\{ & \sum_{\substack{k,l \\ (k \neq l)}} \{P'_{kl}(x)\{B_k(\Lambda_R)\}_{klqp} - P_{kl}(x)\{B_l(\Lambda_R)\}_{klqp}\} \overset{++}{\tilde{\varphi}}{}^{(in)}_{qp} \\ & - \mathrm{i} \sum_k (P'_{kk} - P_{kk})\Delta_k^{-1}(x) \int_x^{\infty} \mathrm{d}y\ \Delta_k(y) \sum_{p \neq k} \sum_{m,q} \{P'_{kp}(y)\{B_k(\Lambda_R)\}_{kpmq} \overset{++}{\tilde{\varphi}}{}^{(in)}_{mq} \\ & - P_{pk}(y)\{B_k(\Lambda_R)\}_{pkmq} \overset{++}{\tilde{\varphi}}{}^{(in)}_{mq}\} \Bigg\} = 0\end{aligned} \tag{2.11}$$

where

$$\Delta(x) = \exp\left(\mathrm{i} \int_{-\infty}^{x} \mathrm{d}y\ (P_D(y) - P'_D(y))\right).$$

Integrating equation (2.11) by parts and changing the order of integration, i.e. making the transition from the operator Λ to the adjoint operator Λ^+, we obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}x \sum_{m,q} \sum_{k,l} \overset{++}{\tilde{\varphi}}{}^{(in)}_{mq}(x)(\{B_k(\Lambda_R^+)\}_{mqkl} P'_{kl}\Delta_k - \{B_l(\Lambda_R^+)\}_{mqkl} P_{kl}\Delta_l) = 0 \tag{2.12}$$

where

$$\begin{aligned}\Lambda^+\varphi = {} & -\mathrm{i}\,\partial\varphi/\partial x + (\varphi P'(x) - P(x)\varphi)_F \\ & - \mathrm{i}\Delta(x) \int_{-\infty}^{x} \mathrm{d}y\ \Delta^{-1}(y)(\varphi(y)P'_F(y) - P_F(y)\varphi(y))_D P'(x) \\ & + \mathrm{i}P(x)\Delta(x) \int_{-\infty}^{x} \mathrm{d}y\ \Delta^{-1}(y)(\varphi(y)P'_F(y) - P_F(y)\varphi(y))_D.\end{aligned} \tag{2.13}$$

Formula (2.12) is a relation between F^+, $F^{+\prime}$, P and P' in the transformations conserving the spectral problem (1.1). Equality (2.12) is satisfied, if the expression in parentheses is equal to zero. Hence, the transformations $P \to P'$ conserving equation (1.1) are of the form

$$\sum_{\substack{k,l \\ (k \neq l)}} (\{B_k(\overset{+}{\Lambda}_R)\}_{mnkl} P'_{kl} \Delta_k - \{B_l(\overset{+}{\Lambda}_R)\}_{mnkl} P_{kl} \Delta_l) = 0, \qquad m \neq n;\ m, n = 1, \ldots, N. \tag{2.14}$$

Recall that $B_k(\lambda)$ are arbitrary entire functions.

Transformation properties of the transition matrix are determined by formula (2.3). The transformation law of S can be represented in a more explicit form. Let us write out equation (2.2) as

$$S' - S = -\mathrm{i}SI + S\{S^{-1}(1-B)S'\}_F \tag{2.15}$$

where

$$I = \int_{-\infty}^{+\infty} \mathrm{d}x\ \{F^{+\prime -1}(P'-P)F^{+}\}_D.$$

Taking into account the relation (see appendix 1)

$$(\overset{++}{\Lambda}_R - \lambda)\tilde{\varphi}_F^{(nn)} = \{P_{TF}(x)\Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{nn} - \Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{nn}P'_{TF}(x)\}_R \tag{2.16}$$

we have

$$\begin{aligned} I_{nn} = {} & \int_{-\infty}^{+\infty} \mathrm{d}x\ \mathrm{Tr}[(P'_T - P_T)(\Lambda_R - \lambda)^{-1}(P_{TFR}(x)\Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{nn} \\ & - \Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{nn}P'_{TFR}(x))] \\ & + \int_{-\infty}^{+\infty} \mathrm{d}x\ \mathrm{Tr}\Big\{(P'_T(x) - P_T(x))\Big[\Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{nn} \\ & - \mathrm{i}\Delta^{-1}(x)\int_x^{\infty} \mathrm{d}y\ [(\Lambda_R - \lambda)^{-1}(P_{TFR}(y)\Delta(+\infty)\Delta^{-1}(y)\overset{++}{\delta}{}^{nn} \\ & - \Delta(+\infty)\Delta^{-1}(y)\overset{++}{\delta}{}^{nn}P'_{TFR}(y))P'_T(y) \\ & - P_T(y)(\Lambda_R - \lambda)^{-1}(P_{TFR}(y)\Delta(+\infty)\Delta^{-1}(y)\overset{++}{\delta}{}^{nn} \\ & - \Delta(+\infty)\Delta^{-1}(y)\overset{++}{\delta}{}^{nn}P'_{TFR}(y))]\Big]\Big\}. \end{aligned} \tag{2.17}$$

Hence, transformation of the elements of the transition matrix is determined by the following relation:

$$\sum_l \Big(\delta_{il} - \sum_{k \neq n} S_{ik}(S^{-1})_{kl}(1 - B_l)\Big)S'_{ln} = (1 - \mathrm{i}I_{nn})S_{in}, \qquad i, n = 1, \ldots, N, \tag{2.18}$$

where I_{nn} is given by formula (2.17).

The fairly complex transformation law becomes simple, if the following equalities are satisfied:

$$\mathrm{i}I_{nn} = 1 - B_n - \{S^{-1}(1-B)S'\}_{nn}, \qquad n = 1, \ldots, N, \tag{2.19}$$

i.e.

$$C = B. \tag{2.20}$$

In this case,

$$S \to S' = B^{-1}SB. \tag{2.21}$$

It should be mentioned that the diagonal elements of the matrix S are invariant under the transformations (2.21). In the general case of equation (2.3)

$$S_{nn} \to S'_{nn} = (C_n/B_n)S_{nn} \qquad (n = 1, \ldots, N). \tag{2.22}$$

The set of all transformations of the form (2.14) is given by the set of all diagonal matrices $B(\lambda)$. Therefore, just like the set of diagonal matrices, transformations (2.14) (conserving the spectral problem (1.1)) form the infinite-dimensional Abelian Lie group B, the 'parameters' of which are arbitrary functions $B_l(\lambda)$ $(l = 1, \ldots, N)$.

3. The general form of integrable equations and Bäcklund transformations

3.1.

The infinite-dimensional group B of transformations (2.14), under which the spectral problem (1.1) is invariant, contains transformations of various types. Let us examine the one-parameter subgroup of the group B, which is given by the matrix

$$B(\lambda) = \exp[-\mathrm{i}(t'-t)Y(\lambda)] \tag{3.1}$$

where $Y(\lambda)$ is an arbitrary diagonal matrix (and $C = B$). As is easy to see, this group is a group of time displacements:

$$S(\lambda, t) \to S'(\lambda, t) = \exp[\mathrm{i}(t'-t)Y(\lambda)]S(\lambda, t)\exp[-\mathrm{i}(t'-t)Y(\lambda)] = S(\lambda, t'). \tag{3.2}$$

Inversion of the mapping $P(x, t) \to S(\lambda, t)$ induces the corresponding transformation $P(x, t) \to P'(x, t) = P(x, t')$. It is of the form

$$\sum_{k,l}\{\exp[-\mathrm{i}(t'-t)Y_k(\Lambda_R^+)]\}_{mnkl}P_{kl}(x, t')\Delta_k - \sum_{k,l}\{\exp[-\mathrm{i}(t'-t)Y_l(\Lambda_R^+)]\}_{mnkl}P_{kl}(x, t)\Delta_l = 0. \tag{3.3}$$

Operators Λ_R^+ are given by formula (2.13), in which one should put $P'(x, t) = P(x, t')$. For the case $N = 2$, relations of such a type were found in Calogero and Degasperis (1976, 1977).

Formula (3.3) determines inexplicitly the evolution of $P(x, t)$ in time $t: P(x, t) \to P(x, t')$. Let us consider the infinitesimal displacement $t \to t' = t + \varepsilon$, $\varepsilon \to 0$:

$$P(x, t) \to P(x, t+\varepsilon) = P(x, t) + \varepsilon\, \partial P(x, t)/\partial t.$$

Then from equation (3.3) we obtain partial differential equations

$$\frac{\partial P_{mn}(x, t)}{\partial t} + \mathrm{i}\left[P_F(x, t), \int_{-\infty}^{x} \frac{\partial P_D(y, t)}{\partial t}\right]_{mn} + \mathrm{i}\sum_{k,l}\{Y_l(L_R^+) - Y_k(L_R^+)\}_{mnkl}P_{kl} = 0$$

$$(m \neq n;\ m, n = 1, \ldots, N) \tag{3.4}$$

where $L^+ = \Lambda^+(P' = P)$, i.e.

$$L^+ \cdot = -\mathrm{i}\frac{\partial}{\partial x} - [P(x), \cdot]_F - \mathrm{i}\left[P_F(x), \int_{-\infty}^{x} \mathrm{d}y [P_F(y), \cdot]_D\right]. \tag{3.5}$$

Correspondingly,

$$\mathrm{d}S(\lambda, t)/\mathrm{d}t = \mathrm{i}[Y(\lambda), S(\lambda, t)]. \tag{3.6}$$

Partial differential equations (3.4) are just the equations integrable by the inverse scattering method with the help of the linear spectral problem (1.1). Using the IST method equations (Gelfand–Levitan–Marchenko equations), one can find a broad class of solutions of equations (3.4) (multi-soliton solutions). At $N = 2$ and $P_D = 0$ we have the equations studied in Ablowitz *et al* (1974). Some concrete equations of the (3.4) type with $N \geqslant 3$ are well known. The model of resonantly interacting wave envelopes (Zakharov and Manakov 1975, Ablowitz and Haberman 1975, Kaup 1976) corresponds to linear functions $Y_l(\lambda)$ ($Y_l(\lambda) = Y_l\lambda$, $l = 1, \ldots, N$) and the multi-component nonlinear Schrödinger equation (Manakov 1975) corresponds to quadratic functions $Y_l(\lambda)$ ($Y_l(\lambda) = Y_l\lambda^2$). Equations of the type (3.4) for $Y_l(\lambda) = Y_l\lambda^{-1}$ will be examined in the next section.

A broader class (than equation (3.4)) of integrable equations appears, if P (as in the case $N = 2$ (Calogero and Degasperis 1976)) depends, in addition to t, on a few variables $\boldsymbol{y}$ of time type. Examining the t- and $\boldsymbol{y}$ ($\delta P = \varepsilon(\partial P/\partial t + \boldsymbol{H}(\lambda, t, \boldsymbol{y})\partial P/\partial \boldsymbol{y})$)-infinitesimal displacement, we obtain from equation (2.14)

$$\begin{aligned}
&\frac{\partial P_{mn}(x, t, \boldsymbol{y})}{\partial t} + \sum_{k,l} \{\boldsymbol{H}(L_R^+, t, \boldsymbol{y})\}_{mnkl} \frac{\partial P_{kl}}{\partial \boldsymbol{y}} + \mathrm{i}\bigg\{\left[P_F(x, t, \boldsymbol{y}), \int_{-\infty}^{x} \mathrm{d}z \frac{\partial P_D(z, t, \boldsymbol{y})}{\partial t}\right] \\
&\qquad + \boldsymbol{H}(L_k^+, t, \boldsymbol{y})\left[P_F(x, t, \boldsymbol{y}), \int_{-\infty}^{x} \mathrm{d}z \frac{\partial P_D(z, t, \boldsymbol{y})}{\partial \boldsymbol{y}}\right]\bigg\}_{mn} \\
&\qquad + \mathrm{i}\sum_{k,l} \{Y_l(L_R^+, t, \boldsymbol{y}) - Y_k(L_R^+, t, \boldsymbol{y})\}_{mnkl} P_{kl} = 0, \\
&\qquad m \neq n;\ m, n = 1, \ldots, N.
\end{aligned} \tag{3.7}$$

Thus the class of equations integrable by means of the linear spectral problem (1.1) is characterised by the integro-differential operator L_R^+ and $N + n - 3$ arbitrary functions $\boldsymbol{H}(\lambda, t, \boldsymbol{y})$, $Y_l(\lambda, t, \boldsymbol{y}) - Y_k(\lambda, t, \boldsymbol{y})$ ($l, k = 1, \ldots, N$) (n is the number of independent variables).

In the particular case

$$Y(\lambda) = \Omega(\lambda) Y,$$

where Y is a constant diagonal matrix and $\Omega(\lambda)$ an arbitrary function, equations (3.4) may be written out in the compact form

$$\begin{aligned}
&\frac{\partial P_F}{\partial t} + \mathrm{i}[P_F, \Gamma]_F - \mathrm{i}\Omega_F(L_R^+)[Y, P_F] = 0, \\
&\Gamma = \int_{-\infty}^{x} \mathrm{d}z \frac{\partial P_D(z, t)}{\partial t},
\end{aligned} \tag{3.8}$$

or

$$\frac{\partial P}{\partial t} - \frac{\partial P_D}{\partial t} + \mathrm{i}[P, \Gamma] - \mathrm{i}\Omega_F(L_R^+)[Y, P] = 0. \tag{3.9}$$

3.2.

Let us now turn our attention to the fact that, by virtue of equation (3.2) (or (3.6)) the diagonal elements of the transition matrix are time independent:

$$\mathrm{d}S_D(\lambda)/\mathrm{d}t = 0. \tag{3.10}$$

Hence, $S_{nn}(\lambda)$ ($n = 1, \ldots, N$) are the generating functionals of the integrals of motion. Expanding $\ln S_D(\lambda)$, as usual, in a series of λ^{-1},

$$\ln S_D(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n} C^{(n)}, \tag{3.11}$$

we obtain the infinite series of the integrals of motion $\{C^{(n)}, n = 0, 1, 2, \ldots, \infty\}$ ($C^{(n)}$ are the diagonal matrices with elements $C_m^{(n)}$, $m = 1, \ldots, N$). Expressions for $C^{(n)}$ in terms of $P(x, t)$ may be found through the use of the procedure proposed in Zakharov and Manakov (1975). Let us present here its somewhat modified form.

Let us represent the fundamental matrix F^+ as follows:

$$F^+(x, \lambda) = R(x, \lambda)E(x, \lambda) \exp\left(\int_x^{\infty} \mathrm{d}y\, \chi(y, \lambda)\right) \tag{3.12}$$

where E is the asymptotic of the linear problem (1.1), $\chi(y, \lambda)$ the diagonal matrix, and the matrix R satisfies the condition $R_D = 1$. From equation (3.12) we have

$$\ln S_D(\lambda) = \int_{-\infty}^{+\infty} \mathrm{d}y\, \chi(y, \lambda). \tag{3.13}$$

Substituting equation (3.12) into equation (1.1), we find

$$\partial R/\partial x + \mathrm{i}\lambda[R, A] - R\chi - \mathrm{i}PR = 0. \tag{3.14}$$

Expanding χ and R in asymptotic series of λ^{-1},

$$\begin{aligned} \chi(x, \lambda) &= \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \chi^{(n)}(x), \\ R(x, \lambda) &= 1 + \sum_{n=1}^{\infty} \frac{1}{\lambda^n} R^{(n)}(x), \end{aligned} \tag{3.15}$$

we obtain the recurrence relations

$$\begin{aligned} &\frac{\partial R^{(n)}}{\partial x} - \mathrm{i}[A, R^{(n+1)}] - \chi^{(n)} - \sum_{p=1}^{n} R^{(p)}\chi^{(n-p)} - \mathrm{i}PR^{(n)} = 0, \qquad n = 1, 2, \ldots, \\ &\qquad -\mathrm{i}[A, R^{(1)}] = \mathrm{i}P + \chi^{(0)}. \end{aligned} \tag{3.16}$$

It follows from equation (3.16) that

$$\chi^{(0)} = -\mathrm{i}P_D, \qquad \chi^{(n)} = -\mathrm{i}(PR_F^{(n)})_D, \qquad n = 1, 2, \ldots, \tag{3.17}$$

and $R_F^{(n)}$ are determined from the recurrence relations

$$\begin{aligned} &\frac{\partial R_F^{(n)}}{\partial x} - \mathrm{i}[A, R_F^{(n+1)}] + \mathrm{i} \sum_{p=1}^{n-1} R_F^{(p)}(PR_F^{(n-p)})_D \\ &\qquad - \mathrm{i}(PR_F^{(n)})_F + \mathrm{i}R_F^{(n)}P_D = 0, \qquad n = 1, 2, \ldots, \\ &\qquad R_F^{(1)} = -P_{FR}. \end{aligned} \tag{3.18}$$

Formulae (3.17) and (3.18) enable us to calculate all integrals of motion, which, by virtue of equations (3.11) and (3.15), are

$$C^{(n)}=\int_{-\infty}^{+\infty} \mathrm{d}x\, \chi^{(n)}(x) \qquad (n=0,1,2,\ldots,). \tag{3.19}$$

Note that for all equations of the class (3.4) the integrals of motion $C^{(n)}$ are of the same form, with an accuracy of the concrete reductions of P.

3.3.

Each concrete equation of the type (3.4) is characterised by a definite matrix $Y(\lambda)$ and, correspondingly, by a definite form of the time dependence (3.2) of the transition matrix. It is easy to prove that transformations (2.14) with matrix B, which is independent of t and y, conserve the form of the time dependence of the matrix S. Hence, they transform the solutions of an equation of the type (3.4) into the solutions of the same equation, i.e. these are the usual (auto-) Bäcklund transformations. The group of Bäcklund transformations contains the group of transformations (2.21) as a subgroup. These transformations do not change the diagonal elements of the transition matrix (and hence, the Hamiltonian) and, therefore, form an infinite-dimensional group of symmetry. It may be shown that the integrals of motion (3.19) are connected just with these groups of symmetry.

We shall refer to the transformations (2.14), which change $S_{nn}(\lambda)$ (i.e. $C\neq B$), as Bäcklund transformations. Similarly to the case $N=2$ (Konopelchenko 1979c), the infinite Abelian group of Bäcklund transformations is a direct product $\mathrm{B_c}\otimes\mathrm{B_d}$ of the infinite-dimensional continuous group $\mathrm{B_c}$ of continual Bäcklund transformations and the infinite discrete group $\mathrm{B_d}$ of soliton Bäcklund transformations. The group $\mathrm{B_c}$ includes transformations which do not change the number of zeros in the diagonal elements of the transition matrix. Soliton Bäcklund transformations are the transformations (2.3) changing the number of zeros in $S_{nn}(x)$ and hence adding one or several solitons to the initial solution. The structure and properties of Bäcklund transformations for $N\geqslant 3$ will be considered in considerable detail elsewhere.

Transformations (2.14) with matrix B, which is dependent on t and (or) y, are the generalised Bäcklund transformations (for the case $N=2$ see Calogero and Degasperis (1976)): they change the form of the time dependence of the matrix S, thereby converting into each other the solutions of different (with different Y_l and $\boldsymbol{H}$) equations of the type (3.7).

Thus we see that the one-parameter groups of time displacements, which generate the partial differential equations of the type (3.4), the symmetry groups of these equations, the groups of Bäcklund transformations and generalised Bäcklund transformations are subgroups of the infinite group of transformations conserving the spectral problem (1.1).

4. Integrable equations with singular dispersion law; relativistically invariant equations

The matrix $Y(\lambda)$ coincides, as is easy to see from equation (3.4), with the dispersion matrix of the linearised equation† (as in the case $N=2$ (Ablowitz *et al* 1974)). For

† Here and below we shall consider equations with two independent variables x, t.

entire functions $Y_l(\lambda)$ the explicit form of integrable equations is found by direct calculation.

In the case of the singular dispersion law (for example, $Y(\lambda)=(\lambda-\lambda_0)^{-n}Y$) we apply the method proposed in Flaschka and Newell (1975).

Let us consider the equation of the form

$$\partial P_F/\partial t + \mathrm{i}[P, \Gamma] - \mathrm{i}(L_R^+ - \lambda_0)_F^{-n}[Y, P] = 0 \tag{4.1}$$

where n is an arbitrary positive integer. In appendix 1 it is shown that

$$(L_R^+ - \lambda)\left[A, \sum_{m=1}^{N} Y_m \frac{\bar{\varphi}_{FT}^{+(mm)}}{S_{mm}}\right] = [P_F(x), Y].$$

Taking into account that

$$\sum_{m=1}^{N} Y_m \frac{\bar{\varphi}_T^{+(mm)}}{S_{mm}} = F^+ Y S_D^{-1} \overset{-1}{F^-} \overset{\text{def}}{=} \Pi(x, t, \lambda), \tag{4.2}$$

we have

$$(L_R^+ - \lambda)^{-1}[Y, P(x, t)] = -[A, \Pi_F(x, t, \lambda)]. \tag{4.3}$$

Hence

$$(L_R^+ - \lambda_0)^{-n}[Y, P(x, t)] = -\frac{1}{(n-1)!}\left[A, \left.\frac{\partial^{n-1}\Pi_F(x, t, \lambda)}{\partial\lambda^{n-1}}\right|_{\lambda=\lambda_0}\right]. \tag{4.4}$$

Thus the equation (4.1) is of the form

$$\frac{\partial P_F}{\partial t} + \mathrm{i}[P, \Gamma] + \frac{\mathrm{i}}{(n-1)!}\left[A, \left.\frac{\partial^{n-1}\Pi(x, t, \lambda)}{\partial\lambda^{n-1}}\right|_{\lambda=\lambda_0}\right] = 0. \tag{4.5}$$

By virtue of the singularity of the dispersion law, it is required (for the case of $N=2$ see Flaschka and Newell (1975)) that

$$S_F(\lambda_0) = 0. \tag{4.6}$$

Therefore

$$\Pi(x, t, \lambda_0) = F^+(x, t, \lambda_0) Y \overset{-1}{F^+}(x, t, \lambda_0). \tag{4.7}$$

The quantity $\Pi(x, t, \lambda)$ satisfies an equation which is easily found from the formula (A1.1) and definitions (4.2). It is of the form

$$\partial\Pi/\partial x = \mathrm{i}\lambda[A, \Pi] + \mathrm{i}[P, \Pi]. \tag{4.8}$$

Solving equation (4.8) with respect to Π and substituting into equation (4.5), one can find an equation which is satisfied by $P(x, t)$.

Let us examine the case when $n=1$ and $\lambda_0 = 0$ in more detail. We have (since $[A, \Pi_D]=0$)

$$\partial P_F/\partial t + \mathrm{i}[P, \Gamma] + \mathrm{i}[A, \Pi(x, t, 0)] = 0, \tag{4.9}$$

$$\partial\Pi(x, t, 0)/\partial x = \mathrm{i}[P(x, t), \Pi(x, t, 0)]. \tag{4.10}$$

By virtue of equation (4.7),

$$\Pi(x, t, 0) = F^+(x, t, 0) Y \overset{-1}{F^+}(x, t, 0) \tag{4.11}$$

and from equation (1.1)

$$P(x, t) = \mathrm{i}F^{+}(x, t, 0)\, \partial \overset{-1}{F^{+}}(x, t, 0)/\partial x. \tag{4.12}$$

Equation (4.10) is satisfied identically, by virtue of equations (4.11) and (4.12), and equation (4.9) is of the form ($F^{+} = F^{+}(x, t, 0)$)

$$\frac{\partial}{\partial t}\left(F^{+}\frac{\partial \overset{-1}{F^{+}}}{\partial x}\right)_{F} + \mathrm{i}\left[F^{+}\frac{\partial \overset{-1}{F^{+}}}{\partial x}, \Gamma\right] + [A, F^{+}Y\overset{-1}{F^{+}}] = 0. \tag{4.13}$$

Equation (4.13) is invariant under Lorentz transformations $x \to x' = \rho x$, $t \to t' = \rho^{-1}x$ (x, t are the cone variables). Also it has the invariant group sense where F^{+} belongs to the local group G, and $\mathrm{i}F^{+}\,\partial \overset{-1}{F^{+}}/\partial x = P$ belongs to the algebra of the local group G.

At $N = 2$ ($P_D = 0$, $P_{21} = -P_{12}$) equation (4.13) is the sine-Gordon equation (Ablowitz *et al* 1974, Flaschka and Newell 1975). For $N \geqslant 3$ it is the generalisation of the sine-Gordon equation to an arbitrary group G, and it was considered in Budagov and Takhtajan (1977), Budagov (1978) and Zakharov and Mikhailov (1978) for the first time. For a full accord with Budagov and Takhtajan (1977) and Budagov (1978) one must put $\overset{-1}{F^{+}}(x, t, 0) = u(x, t) \in \mathrm{G}$:

$$\frac{\partial}{\partial t}\left(u^{-1}\frac{\partial u}{\partial x}\right)_{F} - \left[u^{-1}\frac{\partial u}{\partial x}, \int_{-\infty}^{x} \mathrm{d}y \frac{\partial}{\partial t}\left(u^{-1}\frac{\partial u}{\partial y}\right)_{D}\right] + [A, u^{-1}Yu] = 0. \tag{4.14}$$

In Zakharov and Mikhailov (1978) and Budagov (1978) these equations have been shown to be gauge equivalent to the equations of the principal chiral field equations in a space of G/H (where H is the group of diagonal matrices).

Thus, among the equations integrable by the spectral problem (1.1) there is a broad class of relativistically invariant equations (4.14). Bäcklund transformations for these equations are given by relations (2.14) and the conservation laws by formulae (3.17)–(3.19) (with $P = \mathrm{i}u^{-1}\,\partial u/\partial x$).

Equation (4.14), which is a generalisation of the sine-Gordon equation to the general linear group GL(N), allows the following natural group reductions:

(1) the reduction to the group SU(N):

$$u^{+}(x, t)u(x, t) = 1, \qquad A^{+} = A, \qquad Y^{+} = Y \qquad (P^{+}(x, t) = P(x, t)); \tag{4.15}$$

(2) the reduction to the group SO(N):

$$u_T(x, t)u(x, t) = 1 \qquad (P_T(x, t) = -P(x, t)); \tag{4.16}$$

and $\mathrm{i}A$ and $\mathrm{i}Y$ are arbitrary real diagonal matrices;

(3) the reduction to the group Sp(N) (N is even):

$$u_T(x, t)Ju(x, t) = J, \qquad A_T = JAJ, \qquad Y_T = JYJ \qquad (P_T = JPJ) \tag{4.17}$$

where J is the antisymmetric matrix which may be chosen, for example, in the form $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (1 is the unit matrix of the order of $N/2$).

The reductions (4.15) and (4.16) have been also examined in Budagov and Takhtajan (1977) and Budagov (1978).

The reductions (4.15)–(4.17) also take place for the equations with non-singular matrix $Y(\lambda)$.

For even N the case

$$P = \mathrm{i}\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.18}$$

is of interest, where q is the matrix of the order of $N/2$. Under fulfilment of equation (4.18) the linear spectral problem (1.1) is equivalent to the system ($\psi = \binom{u}{v}$, u and v have $N/2$ components)

$$-\partial^2 v/\partial x^2 + q(x, t)v = \lambda^2 v. \tag{4.19}$$

Equations integrable by means of the spectral problem (4.19) have been considered in Calogero and Degasperis (1977).

In the general case and in the reductions (4.15), (4.16) the Hamiltonian structure of equations (3.8) will be analysed in the next sections. For the reductions (4.17), (4.18) this analysis will be made in a separate paper.

5. Hamiltonian structure of integrable equations

It should be mentioned, first of all, that equations (3.8) are gauge equivalent to the equations which do not include the term $[P, \Gamma]$. Indeed, let us make the transformation $\psi \to \tilde{\psi} = \exp[-\mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, P_D(y)]\psi$. In this case, the spectral problem (1.1) is transformed into the system

$$\partial\tilde{\psi}/\partial x = \mathrm{i}\lambda A\tilde{\psi} + \mathrm{i}\tilde{P}\tilde{\psi}$$

where

$$\tilde{P} = \exp\left(-\mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, P_D(y)\right) P \exp\left(\mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, P_D(y)\right) - P_D$$

$$= \exp\left(-\mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, P_D(y)\right) P_F \exp\left(\mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, P_D(y)\right).$$

Equation (3.8) is converted into the equation

$$\partial\tilde{P}/\partial t - \mathrm{i}\Omega(\tilde{L}_R^+)[Y, \tilde{P}] = 0 \tag{5.1}$$

where the operator $\tilde{L}^+$ is given by formula (3.5), in which one should make the substitution $P \to \tilde{P}$ and take into account that $\tilde{P}_F = \tilde{P}$ ($\tilde{P}_D = 0$). Of course, the results of the preceding sections and appendices (with the simplification $\tilde{P}_D = 0$) are true for the equations of the form (5.1) and operator $\tilde{L}^+$.

We are now going to prove that equations (5.1) (which are gauge equivalent to the equations (3.8)) are Hamiltonian ones for arbitrary N†.

Let us first consider the case of the general position and reduction $P^+ = P$. For the sake of simplicity, for the non-singular dispersion law let us confine ourselves to equations of the form

$$\partial\tilde{P}/\partial t - \mathrm{i}(\tilde{L}_R^+)^n_F[Y, \tilde{P}] = 0 \tag{5.2}$$

where n is an arbitrary positive integer.

From equations (A1.13) and (4.2) we have

$$[A, \tilde{\Pi}(x, t, \lambda)] = (\lambda - \tilde{L}_R^+)^{-1}[Y, \tilde{P}(x, t)]. \tag{5.3}$$

† The Hamiltonian structure of equations (5.1) at $N = 2$ is analysed in Flaschka and Newell (1975) in considerable detail.

Expanding the left-hand side and right-hand side of equation (5.3) in asymptotic series in λ^{-1}, we find

$$\tilde{L}_R^{+n}[Y, \tilde{P}(x, t)] = [A, \tilde{\Pi}^{(n+1)}(x, t)]$$

where

$$\tilde{\Pi}(x, t, \lambda) \sim \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \tilde{\Pi}^{(n)}(x, t). \tag{5.4}$$

On the other hand, it follows from (2.2) that

$$\delta \ln \tilde{S}_{mm} = -\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}x \sum_{k,l} \delta \tilde{P}_{kl} \frac{\bar{\varphi}_{kl}^{+(mm)}}{S_{mm}} \qquad (m = 1, \ldots, N) \tag{5.5}$$

where $\delta \tilde{P}$ is the arbitrary variation of $\tilde{P}$. Hence

$$\tilde{\Pi}(x, t, \lambda) = \mathrm{i} \frac{\delta \operatorname{Tr}(Y \ln \tilde{S}_D(\lambda))}{\delta \tilde{P}_T(x, t)} \tag{5.6}$$

where $\delta/\delta P$ is the variational derivative and Tr denotes the matrix trace. Taking into account equations (5.4) and (3.11), we obtain

$$\tilde{\Pi}^{(n)}(x, t) = \mathrm{i} \frac{\delta \operatorname{Tr}(YC^{(n)})}{\delta \tilde{P}_T(x, t)} \tag{5.7}$$

where $C^{(n)}$ is the diagonal matrix of the integrals of motion.

Hence, equation (5.2) may be written as follows:

$$\partial \tilde{P}/\partial t + [A, \delta H_n/\delta \tilde{P}_T] = 0 \tag{5.8}$$

where $H_n = \operatorname{Tr}(YC^{(n+1)})$. In the case of $P_D = 0$, the analogous result was obtained in Newell (1979).

Equation (5.8) is of the form

$$\partial \tilde{P}/\partial t = \{\tilde{P}, H_n\} \tag{5.9}$$

if one gives the following Poisson brackets ($I(p)$, $H(p)$ are the scalar functionals):

$$\{I, H\} = \int_{-\infty}^{+\infty} \mathrm{d}x \operatorname{Tr}\left(\frac{\delta I}{\delta \tilde{P}}\left[A, \frac{\delta H}{\delta \tilde{P}}\right]\right). \tag{5.10}$$

The quantities $\tilde{P}$ and $\tilde{P}_{TR}$ form a pair of canonical (matrix) variables. The results obtained here are also valid at $P^+ = P$: the pairs of canonical variables form the quantities located symmetrically with respect to the diagonal—they may be considered as the independent ones.

The Poisson bracket (5.10) is not the only bracket corresponding to equation (5.2). Similarly to the case when $N = 2$ (Magri 1978, Kulish and Reiman 1978), the infinite set of symplectic structures is associated with equations of the form (5.1). Let us consider the following Poisson bracket (for $N = 2$ see Kulish and Reiman (1978)):

$$\{I, H\}_m = \int_{-\infty}^{+\infty} \mathrm{d}x \operatorname{Tr}\left(\frac{\delta I}{\delta \tilde{P}_T} L_R^{+m}\left[A, \frac{\delta H}{\delta \tilde{P}_T}\right]\right). \tag{5.11}$$

It is easy to see that equation (5.2) is of the form

$$\partial \tilde{P}/\partial t = \{\tilde{P}, H_{n-m}\}_m \tag{5.12}$$

where m is an arbitrary integer. Hence, the infinite set of Hamiltonian–Poisson bracket pairs corresponds to the concrete equation of the form (5.2).

Let us prove the Hamiltonian structure of equations (5.1) with $\Omega = (\lambda - \lambda_0)^{-n}$. From the relation (5.3) we have

$$(\tilde{L}_R^+ - \lambda_0)^{-n}[Y, \tilde{P}] = \frac{1}{(n-1)!}\left[A, \left.\frac{\partial^{n-1}\tilde{\Pi}(x, t, \lambda)}{\partial\lambda^{n-1}}\right|_{\lambda=\lambda_0}\right]. \tag{5.13}$$

Taking into account equations (5.6), we find

$$\frac{\partial\tilde{P}}{\partial t} + \left[A, \frac{\delta}{\delta\tilde{P}_T}\operatorname{Tr}\left(\frac{1}{(n-1)!}\, Y \left.\frac{\partial^{n-1}\ln\tilde{S}_D(\lambda)}{\partial\lambda^{n-1}}\right|_{\lambda=\lambda_0}\right)\right] = 0. \tag{5.14}$$

Thus, equations (5.1) with $\Omega = (\lambda - \lambda_0)^{-n}$, which are gauge equivalent to equations (4.1), are Hamiltonian ones with the Poisson bracket (5.10) and Hamiltonian

$$H_n = \frac{1}{(n-1)!}\operatorname{Tr}\left(Y \left.\frac{\partial^{n-1}\ln\tilde{S}_D(\lambda)}{\partial\lambda^{n-1}}\right|_{\lambda=\lambda_0}\right). \tag{5.15}$$

In particular, equations (5.1) with $\Omega = \lambda^{-1}$, which are gauge equivalent to the generalisations of the sine-Gordon equation to the groups GL(N) and SU(N) (see § 4)†, are Hamiltonian equations. The Hamiltonian of these equations is of the form

$$H = \operatorname{Tr}[Y \ln\tilde{S}_D(0)]. \tag{5.16}$$

Expression (5.16) may be transformed as follows. From the relation (2.2) we have

$$\tilde{S} - 1 = -\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x\, \exp(-\mathrm{i}\lambda Ax)\tilde{P}(x, t)\tilde{F}^+(x, t, \lambda). \tag{5.17}$$

Hence

$$\ln\tilde{S}_D(0) = \ln[1 - \tilde{F}_D^+(x = +\infty, t, 0) + \tilde{F}_D^+(x = -\infty, t, 0)]. \tag{5.18}$$

Thus the Hamiltonian (5.16) is

$$H = \operatorname{Tr}\{Y \ln[1 + \tilde{u}_D^{-1}(x = -\infty, t) - \tilde{u}_D^{-1}(x = +\infty, t)]\} \tag{5.19}$$

where $\tilde{u}(x, t) = \tilde{F}^{+^{-1}}(x, t, 0)$.

In the combined cases when $Y(\lambda)$ contains the singular and regular parts, the Hamiltonian structure of the equations is proved in a similar way.

It should be mentioned that the integrals of motion $C^{(n)}$ ($n = 1, 2, \ldots$) of equations of the form (5.1) are connected with the form invariance of these equations with respect to the transformations which in the infinitesimal form are the following:

$$\delta_n\tilde{P}(x, t) = -\varepsilon\left[A, \frac{\delta C^{(n)}}{\delta\tilde{P}_T(x, t)}\right] = \mathrm{i}\varepsilon(\tilde{L}_R^+)_F^{n-1}[Y, \tilde{P}(x, t)], \qquad n = 1, 2, \ldots,$$

where ε is the diagonal matrix (of the order of N) of parameters.

† The Hamiltonian structure of equations (4.14) under the group SU(2) was proved in Zakharov and Mikhailov (1978). In our work N is arbitrary. Note also that the transformation $P \to \tilde{P} = \exp(-\mathrm{i}R)P_F\exp(\mathrm{i}R)$ ($R = \int_{-\infty}^{x}\mathrm{d}y\, P_D(y)$), for equations (4.14), which, as we have seen, is the gauge transformation, has been considered in Budagov (1978).

6. The structure of equations at $P_T = -P$

In this case, not all the variables P_{kl} are dynamically independent. For this reason, the analysis of the foregoing sections should be modified.

Let us introduce the upper triangular matrix Q with zeros along the diagonal, such that

$$P = Q - Q_T, \tag{6.1}$$

i.e.

$$Q_{kl} = P_{kl} \qquad \text{for } l > k,$$

$$Q_{kl} = 0 \qquad \text{for } l \leqslant k.$$

Now, let us transform equations (3.8) in such a way that they contain only the independent dynamical variables Q†.

To this end, we return to equations (2.7). They may be written for $B = 1 - \mathrm{i}\varepsilon Y$ and $P' = P + \varepsilon\, \partial P/\partial t$ (see (3.1)–(3.4)) in the following form ($\overset{++}{\varphi}{}^{(in)}_{kl} = (\overset{-1}{F^+})_{ik}(F^+)_{ln}$):

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}\Big[\Big(\frac{\partial P_T}{\partial t} - \mathrm{i}[Y, P]_T \Omega(\lambda)\Big)\overset{++}{\varphi}{}^{(in)}\Big] = 0 \qquad (i \neq n). \tag{6.2}$$

Substituting the definition (6.1) into equation (6.2) and using the properties of the trace, we obtain

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}\Big(\frac{\partial Q_T}{\partial t}(\overset{++}{\varphi}{}^{(in)} - \overset{++}{\varphi}{}^{(in)}_T) - \mathrm{i}[Y, Q]_T \Omega(\lambda)(\overset{++}{\varphi}{}^{(in)} + \overset{++}{\varphi}{}^{(in)}_T)\Big) = 0 \qquad (i \neq n). \tag{6.3}$$

Then let us introduce the projection operations Δ_+ and Δ_-:

$$(Z_{\Delta_+})_{kl} = \begin{cases} Z_{kl} & \text{for } l > k, \\ 0 & \text{for } l < k, \end{cases}$$
$$(Z_{\Delta_-})_{kl} = \begin{cases} 0 & \text{for } l > k, \\ Z_{kl} & \text{for } l < k, \end{cases} \tag{6.4}$$

where Z is an arbitrary matrix with zeros along the diagonal. It is clear that $Z = Z_{\Delta_+} + Z_{\Delta_-}$, $Z_{\Delta_\pm\Delta_\pm} = Z_{\Delta_\pm}$. Since $Q_{\Delta_+} = Q$, $Q_{T\Delta_+} = 0$, $Q_{T\Delta_-} = Q_T$, equation (6.3) is equivalent to the following:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}\Big(\frac{\partial Q_T}{\partial t}\overset{++}{\chi}{}^{(in)}_{\Delta_+} - \mathrm{i}[Y, Q]_T \Omega(\lambda)\overset{++}{\psi}{}^{(in)}_{\Delta_+}\Big) = 0 \tag{6.5}$$

where $\psi = \varphi_F + \varphi_{FT}$, $\chi = \varphi_F - \varphi_{FT}$. Equation (6.5) already contains the independent variables only. Hence, the transition from equation (6.2) to equation (6.5) is the projection onto the subspace of independent dynamical variables.

The first term of equation (6.5) can be converted into the form which contains ψ_{Δ_+} instead of χ_{Δ_+}. Let us define the quantity $W(x, t)$ by the relation

$$\partial Q/\partial t = D^+_{\Delta_+} W \tag{6.6}$$

where D^+ is the 'covariant' derivative:

$$D^+ \cdot = \partial/\partial x - \mathrm{i}[Q - Q_T, \cdot\,]_F + \mathrm{i}[Q - Q_T, \cdot\,]_{FT}. \tag{6.7}$$

† The results of this section are true for the more general case of equations (3.4).

Taking into account that $\partial Q_T/\partial t = D^{+}_{\Delta_-} W_T$, integrating by parts and using formula (A2.6), we find (assuming $W(x=-\infty)=0$)

$$\int_{-\infty}^{+\infty} dx\, \mathrm{Tr}\Big(\frac{\partial Q_T}{\partial t} \overset{++}{\chi}{}^{(in)}_{\Delta_-}\Big) = -\int_{-\infty}^{+\infty} dx\, \mathrm{Tr}(W_T D^{+}_{\Delta_+} \overset{++}{\chi}{}^{(in)}_{\Delta_+})$$

$$= i\lambda \int_{-\infty}^{+\infty} dx\, \mathrm{Tr}(W_T(x,t)[A, \overset{++}{\psi}{}^{(in)}_{\Delta_+}]). \tag{6.8}$$

As a result, equation (6.5) is of the form

$$\int_{-\infty}^{+\infty} dx\, \mathrm{Tr}(W_T[A, \overset{++}{\psi}{}^{(in)}_{\Delta_+}] - [Y, Q]_T \omega(\lambda^2) \overset{++}{\psi}{}^{(in)}_{\Delta_+}) = 0 \qquad (i \neq n) \tag{6.9}$$

where we put $\Omega(\lambda) = \lambda\omega(\lambda^2)$, because for $P_T = -P$ $\Omega(\lambda)$ should be the antisymmetric function λ.

In appendix 2 it is shown that the following relation holds in the subspace stretched over $\overset{++}{\psi}{}^{(in)}_{\Delta_+}$:

$$L^{(Q)}_{\Delta_+ R} \overset{++}{\psi}{}^{(in)}_{\Delta_+} = \lambda^2 \overset{++}{\psi}{}^{(in)}_{\Delta_+} \qquad (i \neq n) \tag{6.10}$$

where

$$L^{(Q)}\psi = -D^{+} D^{-}_{\Delta_+ R}\psi + 2D^{+}\Big[Q, \int_x^{\infty} dy\, [Q - Q_T, \psi]_D\Big]_R. \tag{6.11}$$

Therefore, equation (6.9) may be written as

$$\int_{-\infty}^{+\infty} dx\, \mathrm{Tr}(W_T[A, \overset{++}{\psi}{}^{(in)}_{\Delta_+}] - [Y, Q]_T \omega(L^{(Q)}_{\Delta_+ R}) \overset{++}{\psi}{}^{(in)}_{\Delta_+}) = 0 \qquad (i \neq n). \tag{6.12}$$

Finally, making the transition from the operator $L^{(Q)}_{\Delta_+}$ to the adjoint operator $L^{(Q)+}_{\Delta_+}$ and taking into account the equality $\mathrm{Tr}(W_T[A, \psi_{\Delta_+}]) = \mathrm{Tr}(\psi_{T\Delta_-}[A, W])$, we obtain

$$\int_{-\infty}^{+\infty} dx\, \mathrm{Tr}\{\overset{++}{\psi}{}^{(in)}_{T\Delta_-}([A, W] - \omega(L^{(Q)+}_{\Delta_+ R})[Y, Q])\} = 0 \tag{6.13}$$

where

$$L^{(Q)+}\cdot = -D^{-} D^{+}_{\Delta_+ R}\cdot - 2\Big[Q(x), \int_{-\infty}^{x} dy\, [Q - Q_T, D^{+}_{\Delta_+ R}\cdot]_D\Big]. \tag{6.14}$$

Equality (6.13) is satisfied if

$$[A, W] - \omega(L^{(Q)+}_{\Delta_+ R})[Y, Q] = 0.$$

Using (6.6), we obtain the following differential equations ($D \overset{\mathrm{def}}{=} D^{+}$):

$$\partial Q/\partial t - D_{\Delta_+}\omega_R(L^{(Q)+}_{\Delta_+ R})[Y, Q] = 0. \tag{6.15}$$

There is no difficulty in seeing that equation (6.15), which contains Q only, is equivalent to equation (3.8) for $P_T = -P$. At $N=2$ $D = \partial/\partial x$ and equation (6.15) coincides with equations considered in Flaschka and Newell (1975).

For singular functions $\omega(\lambda^2)$ of the type

$$\omega(\lambda^2) = (\lambda^2 - \lambda_0^2)^{-n} \tag{6.16}$$

we use the analogue of formula (4.3), which in our case is of the form (see appendix 2)

$$\tfrac{1}{2}[A, \Pi_Q(x,t,\lambda)/\lambda] = (L^{(Q)+}_{\Delta_+R} - \lambda^2)^{-1}[Y, Q] \tag{6.17}$$

where

$$\Pi_Q(x,t,\lambda) = \sum_{m=1}^{N} Y_m \frac{\overset{-+}{\chi}{}^{(mm)}_{\Delta_+}}{S_{mm}}. \tag{6.18}$$

Hence

$$(L^{(Q)+}_{\Delta_+R} - \lambda_0^2)^{-n}[Y, Q] = -\frac{1}{2(n-1)!}\left[A, \left.\frac{\partial^{n-1}(\Pi_Q(x,t,\lambda)/\lambda)}{\partial(\lambda^2)^{n-1}}\right|_{\lambda=\lambda_0}\right]. \tag{6.19}$$

Thus the equations with the dispersion law (6.16) are of the form

$$\frac{\partial Q}{\partial t} + \frac{1}{2(n-1)!} D_{\Delta_+} \left.\frac{\partial^{n-1}(\Pi_Q(x,t,\lambda)/\lambda)}{\partial(\lambda^2)^{n-1}}\right|_{\lambda=\lambda_0} = 0. \tag{6.20}$$

7. The Hamiltonian structure of equations for $P_T = -P$

Equations of the type (6.15) contain only the dynamical independent quantities and admit the natural Hamiltonian structure.

Let us first consider the equations of the form

$$\partial Q/\partial t - \mathrm{i} D_{\Delta_+}(L^{(Q)+}_{\Delta_+R})^n_R[Y, Q] = 0. \tag{7.1}$$

From equation (2.2) (see also (5.5)) we have

$$\delta \ln S_{mm} = -\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}\left(Q_{T\Delta_-} \frac{\overset{-+}{\chi}{}^{(mm)}_{\Delta_+}}{S_{mm}}\right). \tag{7.2}$$

Hence

$$\Pi_Q(x,t,\lambda) = \mathrm{i}\frac{\delta}{\delta Q(x,t)} \mathrm{Tr}[Y \ln S_D(\lambda)]. \tag{7.3}$$

Expanding the left- and right-hand parts of equation (6.17) in an asymptotic series in λ^{-1}, we find

$$(L^{(Q)+}_{\Delta_+R})^n[Y, Q] = -\tfrac{1}{2}[A, \Pi_Q^{(2n+1)}].$$

Taking into account equations (7.3) and (3.11), we obtain

$$(L^{(Q)+}_{\Delta_+R})^n[Y, Q] = -\frac{\mathrm{i}}{2}\left[A, \frac{\delta}{\delta Q}\mathrm{Tr}(YC^{(2n+1)})\right]. \tag{7.4}$$

Hence equation (7.1) may be written as follows:

$$\frac{\partial Q}{\partial t} - \tfrac{1}{2} D_{\Delta_+} \frac{\delta}{\delta Q} \mathrm{Tr}(YC^{(2n+1)}) = 0. \tag{7.5}$$

It is obvious that equation (7.5) is a Hamiltonian one. The Poisson bracket is of the form

$$\{I, H\} = \int_{-\infty}^{+\infty} \mathrm{d}x\, \mathrm{Tr}\left(\frac{\delta I}{\delta Q_T} D_{\Delta_+} \frac{\delta H}{\delta Q}\right) \tag{7.6}$$

and the Hamiltonian is equal to

$$H_n = \tfrac{1}{2}\,\mathrm{Tr}(YC^{(2n+1)}). \tag{7.7}$$

It is clear that for the equation

$$\partial Q/\partial t - \mathrm{i}D_{\Delta_+}\omega_R(L^{(Q)+}_{\Delta_+R})[Y, Q] = 0 \tag{7.8}$$

where $\omega(\lambda^2) = \Sigma_\alpha\,\omega_\alpha(\lambda^2)^\alpha$, the Hamiltonian H_ω is equal to

$$H_\omega = \tfrac{1}{2}\,\mathrm{Tr}\Big(Y\sum_\alpha \omega_\alpha C^{(2\alpha+1)}\Big). \tag{7.9}$$

Just as in the case of the general situation (see § 5), the infinite number of Hamiltonian–Poisson bracket pairs is connected with the equations of the form (7.8).

Let us proceed now to the equations with the singular dispersion law. Let us prove the Hamiltonian structure of the equation

$$\partial Q/\partial t - D_{\Delta_+}(L^{(Q)+}_{\Delta_+R})^{-1}_R[Y, Q] = 0. \tag{7.10}$$

The case $\omega = (\lambda^2)^{-n}$ is analysed in a similar way. From equation (6.17) we have that

$$\frac{\partial Q}{\partial t} - \frac{1}{2}D_{\Delta_+}\bigg[A, \frac{\Pi_Q(x, t, \lambda)}{\lambda}\bigg|_{\lambda=0}\bigg]_R = 0. \tag{7.11}$$

By virtue of equation (7.3),

$$\begin{aligned}\frac{\Pi_Q(x, t, \lambda)}{\lambda}\bigg|_{\lambda=0} &= \Big(\frac{\partial}{\partial\lambda}\Pi_Q(x, t, \lambda)\Big)\bigg|_{\lambda=0}\\ &= \mathrm{i}\frac{\delta}{\delta Q}\mathrm{Tr}\bigg[Y\Big(\frac{\partial}{\partial\lambda}\ln S_D(\lambda)\Big)\bigg|_{\lambda=0}\bigg].\end{aligned} \tag{7.12}$$

Hence equation (7.10) is of the form

$$\partial Q/\partial t = \{Q, H\} \tag{7.13}$$

where

$$H = \frac{\mathrm{i}}{2}\mathrm{Tr}\Big(Y\frac{\partial}{\partial\lambda}\ln S_D(0)\Big) \tag{7.14}$$

and the Poisson bracket is given by formula (7.6).

It is easy to prove that equation (7.10) is equivalent to the SO(N)-Gordon equation (the sine-Gordon equation under the group SO(N)) which has been considered in § 4. Indeed, taking into account (by virtue of $S_F(\lambda = 0) = 0$) that

$$\Pi_Q = \sum_{m=1}^{N} Y_m \overset{++}{\chi}{}^{(mm)}_{\Delta_+}$$

and using (A2.4), we find

$$\partial Q/\partial t + \tfrac{1}{2}\mathrm{i}[A, \Pi_{T\Delta_+} + \Pi_{\Delta_+}] = 0 \tag{7.15}$$

where Π is given by formula (4.7). Transposing (7.15) and subtracting the resulting equation from (7.15), we derive equation (4.9) where $P = Q - Q_T$.

Thus we have shown that the SO(N)-Gordon equation at arbitrary N is a Hamiltonian one with the Poisson bracket (7.6). Let us reduce the Hamiltonian (7.14) of this equation to a more explicit form. For this purpose, we use the identity

$$\mathrm{i}\overset{-1}{F^{-}}(x, t, \lambda)AF^{+}(x, t, \lambda)-\mathrm{i}AS=\frac{\partial}{\partial x}\Big(-\frac{\partial \overset{-1}{F^{-}}(x, t, \lambda)}{\partial \lambda}F^{+}(x, t, \lambda)\Big)-\mathrm{i}AS \tag{7.16}$$

which is obtained directly from (1.1).

We find from equation (7.16) that

$$\frac{\partial}{\partial \lambda}\ln S_{mm}=-\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}x\,\Big((\overset{-1}{F^{-}}AF^{+})_{mm}\frac{1}{S_{mm}}-A_{mm}\Big) \qquad (m=1,\ldots,N). \tag{7.17}$$

Hence (taking into account that $S_F(\lambda=0)=0$)

$$\begin{aligned} H&=\frac{\mathrm{i}}{2}\sum_{m=1}^{N}Y_m\frac{\partial}{\partial\lambda}\ln S_{mm}(0)\\ &=\tfrac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{Tr}(AF^{+}(x, t, 0)\,Y\overset{-1}{F^{+}}(x, t, 0)-AY). \end{aligned} \tag{7.18}$$

Denoting $\overset{-1}{F^{+}}(x, t, 0)=u(x, t)$ (see § 4), we have

$$H=\tfrac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{Tr}(Au_TYu-AY). \tag{7.19}$$

In this notation

$$Q=\mathrm{i}(u_T\,\partial u/\partial x)_{\Delta_+}. \tag{7.20}$$

As we have already noted, the Hamiltonian structure of the SO(3)-Gordon equation has been proved in Budagov and Takhtajan (1977) and Budagov (1978). The Hamiltonian in those papers coincides (with an accuracy of the transition to cone variables) with the Hamiltonian (7.19). However, the canonical variables are different. In Budagov and Takhtajan (1977) and Budagov (1978), the canonical variables are natural coordinates of the local group SO(3). In our case (as is seen from equation (7.20)), the natural coordinates in the local algebra SO(N) play the role of canonical coordinates. The situation for the GL(N)-Gordon and SU(N)-Gordon equations is similar (see § 5).

Note that by virtue of the gauge equivalence of the G-Gordon equations to the equations of the principal chiral field over the space of flags (Budagov 1978, Zakharov and Mikhailov 1978), the latter are Hamiltonian ones.

In conclusion, it should be mentioned that the Lagrangian structure of some equations connected with the principal chiral field, namely the four-fermion type equations, has been proved in Zakharov and Mikhailov (1980).

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Appendix 1.

In this appendix we shall obtain some relations which include the quantities $\tilde{\varphi}_{kl}^{(in)}=(\psi^{-1})_{ik}(\psi')_{ln}$ and integro-differential operators Λ, Λ^{+}, L, L^{+}.

We shall use the following notation: $\overset{++}{\tilde{\varphi}}{}^{(in)}_{kl} = (F^{+})^{-1}_{ik}(F^{+\prime})_{ln}$, $\overset{-+}{\tilde{\varphi}}{}^{(in)}_{kl} = (F^{-})^{-1}_{ik}(F^{+\prime})_{ln}$. From equation (1.1) we find

$$\partial\tilde{\varphi}^{(in)}/\partial x = -\mathrm{i}\lambda[A, \tilde{\varphi}^{(in)}] + \mathrm{i}\tilde{\varphi}^{(in)}P'_T - \mathrm{i}P_T\tilde{\varphi}^{(in)}. \tag{A1.1}$$

Hence

$$\partial\tilde{\varphi}^{(in)}_D/\partial x = \mathrm{i}(P'_D - P_D)\tilde{\varphi}^{(in)}_D + \mathrm{i}(\tilde{\varphi}^{(in)}_F P'_T - P_T\tilde{\varphi}^{(in)}_F)_D. \tag{A1.2}$$

From equation (A1.2) we have

$$\begin{aligned}\tilde{\varphi}^{(in)}_D(x) = {} & \Delta(+\infty)\Delta^{-1}(x)\tilde{\varphi}^{(in)}_D(+\infty) \\ & -\mathrm{i}\Delta^{-1}(x)\int_x^\infty \mathrm{d}y\ \Delta(y)(\tilde{\varphi}^{(in)}_F(y)P'_T(y) - P_T(y)\tilde{\varphi}^{(in)}_F(y))_D\end{aligned} \tag{A1.3}$$

or

$$\begin{aligned}\tilde{\varphi}^{(in)}_D(x) = {} & \Delta(+\infty)\Delta^{-1}(x)\tilde{\varphi}^{(in)}_D(-\infty) \\ & +\mathrm{i}\Delta^{-1}(x)\int_{-\infty}^x \mathrm{d}y\ \Delta(y)(\tilde{\varphi}^{(in)}_F(y)P'_T(y) - P_T(y)\tilde{\varphi}^{(in)}_F(y))_D\end{aligned} \tag{A1.4}$$

where

$$\Delta(x) = \exp\left(\mathrm{i}\int_{-\infty}^x \mathrm{d}y\ (P_D(y) - P'_D(y))\right).$$

Following from the asymptotic properties of F^+ and F^-, we obtain

$$\begin{aligned}(\overset{++}{\tilde{\varphi}}{}^{(in)}_D(+\infty))_{kk} &\overset{\text{def}}{=} (\overset{++}{\delta}{}^{in})_{kk} = \delta_{ik}\delta_{kn}, \\ (\overset{-+}{\tilde{\varphi}}{}^{(in)}_D(-\infty))_{kk} &\overset{\text{def}}{=} (\overset{-+}{\delta}{}^{in})_{kk} = \delta_{ik}S_{kn}.\end{aligned} \tag{A1.5}$$

From equations (A1.1), (A1.3), and (A1.5) we obtain

$$\Lambda\overset{++}{\tilde{\varphi}}{}^{(in)}_F = \lambda[A, \overset{++}{\tilde{\varphi}}{}^{(in)}_F] + P_{TF}(x)\Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{in} - \Delta(+\infty)\Delta^{-1}(x)\overset{++}{\delta}{}^{in}P'_{TF}(x) \tag{A1.6}$$

where

$$\begin{aligned}\Lambda\varphi = {} & \mathrm{i}\,\partial\varphi/\partial x + (\varphi P'_T(x) - P_T(x)\varphi)_F \\ & -\mathrm{i}\Delta^{-1}(x)\int_x^\infty \mathrm{d}y\ \Delta(y)(\varphi(y)P'_T(y) - P_T(y)\varphi(y))_D P'_{TF}(x) \\ & +\mathrm{i}P_{TF}(x)\Delta^{-1}(x)\int_x^\infty \mathrm{d}y\ \Delta(y)(\varphi(y)P'_T(y) - P_T(y)\varphi(y))_D.\end{aligned} \tag{A1.7}$$

In particular,

$$\Lambda\overset{++}{\tilde{\varphi}}{}^{(in)}_F = \lambda[A, \overset{++}{\tilde{\varphi}}{}^{(in)}_F] \qquad (i \neq n) \tag{A1.8}$$

i.e.

$$\Lambda_R\overset{++}{\tilde{\varphi}}{}^{(in)}_F = \lambda\overset{++}{\tilde{\varphi}}{}^{(in)}_F \qquad (i \neq n). \tag{A1.9}$$

There is no difficulty in seeing that for the operator

$$L^+ \cdot \overset{\text{def}}{=} \Lambda^+(P'=P) = -\mathrm{i}\,\partial/\partial x - [P(x), \cdot\,]_F - \mathrm{i}\left[P_F(x), \int_{-\infty}^{x} \mathrm{d}y\, [P_F(y), \cdot\,]_D\right] \tag{A1.10}$$

the following relation holds ($\overset{-+}{\varphi}{}^{(in)}_{kl} = (F^{-^{-1}})_{ik}(F^+)_{ln}$):

$$L^+ \overset{-+}{\varphi}{}^{(in)}_{TF} = \lambda[A, \overset{-+}{\varphi}{}^{(in)}_{TF}] + [P_F(x), \overset{-+}{\delta}{}^{in}]. \tag{A1.11}$$

Hence

$$L^+ \overset{-+}{\varphi}{}^{(nn)}_{TF}/S_{nn} = \lambda[A, \overset{-+}{\varphi}{}^{(nn)}_{TF}/S_{nn}] + [P_F(x), \overset{-+}{\tilde{\delta}}{}^{nn}] \tag{A1.12}$$

where $(\overset{-+}{\tilde{\delta}}{}^{nn})_{kk} = \delta_{kn}$.

Multiplying the equality (A1.12) by Y_n and summing over n, we obtain

$$L^+ \sum_{n=1}^{N} Y_n \frac{\overset{-+}{\varphi}{}^{(nn)}_{TF}}{S_{nn}} = \lambda\left[A, \sum_{n=1}^{N} Y_n \frac{\overset{-+}{\varphi}{}^{(nn)}_{TF}}{S_{nn}}\right] + [P_F(x), Y]. \tag{A1.13}$$

Appendix 2.

Let us consider equation (A1.6) at $P'=P$ and $P_T = -P = -(Q-Q_T)$. We use the following notation: $\overset{\pm+}{\varphi}{}^{(in)}_{kl} = (F^{\pm^{-1}})_{ik}(F^+)_{ln}$ and $\psi = \varphi_F + \varphi_{FT}$, $\chi = \varphi_F - \varphi_{FT}$. It is easy to prove that (for $\varphi = \overset{++}{\varphi}$)

$$\mathrm{i}\,\partial\psi/\partial x + [Q-Q_T, \psi]_F - \mathrm{i}\left[Q-Q_T, \int_{x}^{\infty} \mathrm{d}y\, [Q-Q_T, \psi]_D\right] = \lambda[A, \chi] - 2[Q-Q_T, \overset{++}{\delta}], \tag{A2.1}$$

$$\mathrm{i}\,\partial\chi/\partial x + [Q-Q_T, \chi]_F = \lambda[A, \psi]. \tag{A2.2}$$

Let us apply the operation Δ_+ to equations (A2.1) and (A2.2). We have the following result:

$$\begin{aligned}&\mathrm{i}\,\partial\psi_{\Delta_+}/\partial x + [Q-Q_T, \psi_{\Delta_+}]_{F\Delta_+} + [Q-Q_T, \psi_{\Delta_+}]_{TF\Delta_+}\\ &\qquad -2\mathrm{i}\left[Q, \int_{x}^{\infty} \mathrm{d}y\, [Q-Q_T, \psi_{\Delta_+}]_D\right] = \lambda[A, \chi_{\Delta_+}] - 2[Q, \overset{++}{\delta}],\end{aligned} \tag{A2.3}$$

$$\mathrm{i}\,\partial\chi_{\Delta_+}/\partial x + [Q-Q_T, \chi_{\Delta_+}]_{F\Delta_+} - [Q-Q_T, \chi_{\Delta_+}]_{TF\Delta_+} = \lambda[A, \psi_{\Delta_+}]. \tag{A2.4}$$

In expressions of the type $Z_{TF\Delta_+\ldots}$ the operations are performed from left to right. Note also some obvious but useful properties in calculations: operations T, F, D commute with each other; $T\Delta_+ = \Delta_- T$; $[P_{\Delta_\pm}, Z_{\Delta_\pm}]_{\Delta_\mp} = 0$; for the symmetric matrix φ we have $\varphi = \varphi_{\Delta_+} + \varphi_{\Delta_+ T}$, for the antisymmetric one χ, $\chi = \chi_{\Delta_+} - \chi_{\Delta_+ T}$ and so on.

Let us introduce the 'covariant' derivatives D^-, D^+:

$$D^\pm \cdot = \partial/\partial x - \mathrm{i}[Q-Q_T, \cdot\,]_F \pm \mathrm{i}[Q-Q_T, \cdot\,]_{TF}. \tag{A2.5}$$

Then equation (A2.4) is of the form

$$\mathrm{i}D^{+}_{\Delta_+}\chi_{\Delta_+} = \lambda[A, \psi_{\Delta_+}]. \tag{A2.6}$$

Applying the operation R to equation (A2.3), acting on the resulting equation $D^{+}_{\Delta_+}$ and taking into account (A2.6), we find

$$L^{(Q)}_{\Delta_+}\overset{++}{\psi}{}^{(in)}_{\Delta_+} = \lambda^2[A, \overset{++}{\psi}{}^{(in)}_{\Delta_+}] - 2\mathrm{i}D^{+}_{\Delta_+}[Q_R, \overset{++}{\delta}{}^{in}] \tag{A2.7}$$

where

$$L^{(Q)}\psi = D^{+}\left(-D^{-}_{\Delta_+R}\psi + 2\left[Q(x), \int_x^{\infty} \mathrm{d}y\,[Q - Q_T, \psi(y)]_D\right]_R\right). \tag{A2.8}$$

Hence

$$L^{(Q)}_{\Delta_+R}\overset{++}{\psi}{}^{(in)}_{\Delta_+} = \lambda^2\overset{++}{\psi}{}^{(in)}_{\Delta_+} \qquad (i \neq n). \tag{A2.9}$$

Equations for $\overset{-+}{\psi}_{\Delta_+}$ and $\overset{-+}{\chi}_{\Delta_+}$ are the following:

$$\mathrm{i}D^{-}_{\Delta_+}\overset{-+}{\psi}{}^{(in)}_{\Delta_+} + 2\mathrm{i}\left[Q(x), \int_{-\infty}^{x} \mathrm{d}y\,[Q(y) - Q_T(y), \overset{-+}{\psi}{}^{(in)}_{\Delta_+}(y)]_D\right] = \lambda[A, \overset{-+}{\chi}{}^{(in)}_{\Delta_+}] - 2[Q, \overset{-+}{\delta}{}^{in}], \tag{A2.10}$$

$$\mathrm{i}D^{+}_{\Delta_+}\overset{-+}{\chi}{}^{(in)}_{\Delta_+} = \lambda[A, \overset{-+}{\psi}{}^{(in)}_{\Delta_+}]. \tag{A2.11}$$

With the use of equations (A2.10) and (A2.11) we obtain

$$\begin{aligned} -D^{-}_{\Delta_+}D^{+}_{\Delta_+R}\overset{-+}{\chi}{}^{(in)}_{\Delta_+} - 2\left[Q(x), \int_{-\infty}^{x} \mathrm{d}y\,[Q - Q_T, D^{+}_{\Delta_+R}\overset{-+}{\chi}{}^{(in)}_{\Delta_+}]_D\right] \\ = \lambda^2[A, \overset{-+}{\chi}{}^{(in)}_{\Delta_+}] - 2\lambda[Q, \overset{-+}{\delta}{}^{in}]. \end{aligned} \tag{A2.12}$$

Comparison of the left-hand side of equation (A2.12) and equation (6.14) gives

$$L^{(Q)+}_{\Delta_+}\overset{-+}{\chi}{}^{(in)}_{\Delta_+} = \lambda^2[A, \overset{-+}{\chi}{}^{(in)}_{\Delta_+}] - 2\lambda[Q, \overset{-+}{\delta}{}^{in}]. \tag{A2.13}$$

Hence

$$(L^{(Q)+}_{\Delta_+R} - \lambda^2)[A, \Pi_Q(x, t, \lambda)] = 2\lambda[Y, Q]. \tag{A2.14}$$

References

Ablowitz M J and Haberman R 1975 *J. Math. Phys.* **16** 2301
Ablowitz M S, Kaup D S, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
Budagov A S 1978 *Notes of LOMI Scientific Seminars* **77** 24
Budagov A S and Takhtajan L A 1977 *Dokl. Akad. Nauk SSSR* **235** 805
Bullough R and Caudrey P (ed.) 1980 *Solitons* (Berlin: Springer)
Calogero F (ed.) 1978 *Research Notes in Mathematics* **26** (London: Pitman)
Calogero F and Degasperis A 1976 *Nuovo Cimento* **32B** 201
—— 1977 *Nuovo Cimento* **39B** 1

Flaschka H and Newell A C 1975 *Lecture Notes in Physics* **38** 355
Gerdjikov V S, Ivanov M I and Kulish P P 1979 *JINR Preprint* E2-12590 Dubna
Kaup D J 1976 *Stud. Appl. Math.* **55** 9
Konopelchenko B G 1979a *Preprint* INP 79-82 Novosibirsk
—— 1979b *Preprint* INP 79-135 Novosibirsk
—— 1979c *Phys. Lett.* **74A** 189
Kulish P P 1979 *Preprint LOMI* P-3-79 Leningrad (in Russian)
Kulish P P and Reiman L G 1978 *Notes of LOMI Scientific Seminars* **77** 134
Magri F 1978 *J. Math. Phys.* **19** 1156
Manakov S V 1975 *Zh. Eksp. Teor. Fiz.* **65** 505
Newell A C 1979 *Proc. R. Soc.* A **365** 283
Scott A C, Chu F Y F and McLaughlin D W 1973 *Proc. IEEE* **61** 1433
Zakharov V E and Manakov S V 1975 *Zh. Eksp. Teor. Fiz.* **69** 1654
Zakharov V E and Mikhailov A V 1978 *Zh. Eksp. Teor. Fiz.* **74** 1953
—— 1980 *Functional analysis and its application* **14** 55
Zahkarov V E and Shabat A B 1974a *Functional analysis and its application* **8** 43
—— 1974b *Functional analysis and its application* **13** 13 (in Russian)